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# ON THE OPTIMAL CHOICE OF UNKNOWNS IN RATIO-TYPE ESTIMATORS

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SUMMARY

The optimal estimators of unknown population parameters in the several classes of ratio (product)-type estimators studied by various authors depend on some functions of unknown parameters. The general result proved in this paper shows that, in a 'regular class' of estimators (which includes almost all classes of estimators considered by various authors), the replacement of some or all of these parameters by their unbiased estimates does not alter the efficiency of the optimal estimators (to the first order of approximation) in terms of its mean square error.

*Keywords* : Ratio type estimators, Mean square error, Auxiliary variable.

## Introduction

In recent years many authors have proposed ratio (product)-type estimators for the population mean $\overline{Y}$ of the study variable $y$ using the auxiliary information $x$. For a detailed list of such estimators one can refer to Singh [2]. Even though such estimators are more efficient than the conventional ratio (product) estimator, they require the knowledge of some function of population values of both the study as well as auxiliary variables link $K = \overline{Y} S_{xy} \{\overline{X} S_x^2\}^{-1}$. Srivastava *et al.* [6] have considered the problem of estimation of these optimum values. The optimality of the underlying estimator when the unknown quantities are replaced by their estimated values has not been studied in that paper. In one of their earlier papers Srivastava and Jhajj [5] have proved that in a general class of estimators when the unknown functions are replaced by their consistent estimators the resulting estimator will have the same mean square error upto terms of order $n^{-1}$ as that of the original estimator under certain conditions (see Theorem 3 of Srivastava and Jhajj [5]).

In this paper, it is proposed to indentify the class of estimators which when the unknown quantities in the estimators are replaced by their unbiased estimators the mean square error remains the same (to the first order of approximation).

## 2. Regular Class

In general the ratio (product)-type estimators proposed by several authors involve functions of sample values of the study variable and functions of population as well as the sample values of the auxiliary variable. To be specific, let the estimator for $\bar{Y}$ be of the from

$$\hat{\bar{Y}}_\alpha = g(t_1, t_2, \ldots, t_p, \alpha) \tag{2.1}$$

where $t_1, t_2, \ldots, t_p$ are functions of sample (of both study as well as auxiliary variables) values and $\alpha$ is the unknown constant or a vector of constants whose optimal value(s) (in the mean square error sense) can be function(s) of population values of study as well as auxiliary variables. Let $e_i = (t_i - T_i)/T_i$, $i = 1, 2, \ldots, p$ where $E(t_i) = T_i$, $i = 1, \ldots, p$. The probabilistic properties of $e_1, \ldots, e_p$ are exploited to study the bias and mean square error of the estimator on replacing $t_i$ by $T_i(1 + e_i)$, $i = 1, \ldots, p$.

*Definition* 2.1. An estimator $\hat{\bar{Y}}_\alpha$ is a regular estimator for $\bar{Y}$ if it can be written in the form

$$\hat{\bar{Y}}_\alpha = \bar{Y} + f(\underline{\alpha}, e_1, \ldots, e_p) \tag{2.2}$$

wher $f(., \ldots, .)$ is a polynomial in $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)'$ and $e_1, \ldots, e_p$ with o term independent of $e$'s.

*Definition* 2.2. The collection of all regular estimators is called regular class.

It may be noted that the estimators in the classes considered by Srivastava [3], Srivastava and Jhajj [4] [5] are members of the regular class. The following examples make this idea clear.

EXAMPLE 2.1 : The estimator

$$\bar{Y}_\alpha^{(1)} = \bar{y}(\bar{x}/\bar{X})^\alpha$$

is a member of the Srivastava (1971) class. It can be written as

$$\bar{Y}_\alpha^{(1)} = Y + g\,(\alpha, e_1, e_2)$$

where $e_1 = (\bar{x} - \bar{X})/\bar{X}$ and $e_2 = (\bar{y} - \bar{Y})/\bar{Y}$.

EXAMPLE 2.2 : The estimator

$$\bar{y}_{\alpha\beta}^{(2)} = \bar{y}\,(\bar{x}/\bar{X})^\alpha\,(s_x^2/S_x^2)^\beta$$

is a member of the Srivastava and Jhajj (1981) class of estimators. Clearly $\bar{Y}_{\alpha\beta}^{(2)} = f\,(\underline{\alpha}, t_1, t_2, t_3)$, where $t_1 = \bar{y}$, $t_2 = \bar{x}$, $t_3 = s_x^2$ and $\underline{\alpha} = (\alpha, \beta)'$. Taking $e_1 = (\bar{x} - \bar{X})/\bar{X}$, $e_2 = (\bar{y} - \bar{Y})/\bar{Y}$ and $e_3 = (s_x^2 - S_x^2)/S_x^2$ it can be seen that this estimator is of the form (2.2).

The following is an example of an estimator which is not a member of any of the above classes but a member of the regular class defined in (2.2).

EXAMPLE 2.3 :

$$Y^{(3)} = \{\bar{y} + (\bar{X} - \bar{x})\}\,(\bar{x}/\bar{X})^\alpha$$

This estimator is due to Ray and Singh [1].

In order to make the discussion simple, we assume that the estimator contains only one unknown quantity. It may be noted that to the first order of approximation the mean square error of $\hat{Y}_a$ defined in (2.1) will be always of the form

$$\sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij}(\alpha)\, E\,(e_i\, e_j) \qquad (2.3)$$

where $a_{ij}(\alpha) = b_{ij}\,\alpha^k$, $k$ is a non-negative integer and $b_{ij}$'s are known constants.

Let $\alpha_o$ be the unknown constant involved in the estimator $\hat{Y}_\alpha$ whose optimal value has the representation

$$\alpha_0 = \left\{\prod_{s=1}^{n} c_s\,\lambda_s\right\}^{-1} \sum_{i=1}^{k}\left\{\prod_{j=1}^{m_i} d_{ij}\,\theta_{ij}\right\}\left\{\prod_{r=1}^{n_i} f_{ir}\gamma_{ir}\right\}^{-1} \tag{2.4}$$

where $\theta_{ij}, \gamma_{ij}$ and $\lambda_s$ are parametric quantities which are estimable unbiasedly and $d_{ij}$, $c_s$, and $f_{ir}$ are known scalars.

A theorem concerning the regular class when the optimal value of the unknown quantity in the underlying estimator has the representation given in (2.4) is established below :

THEOREM 2.1 : *If $\hat{\alpha}_0$ is the value of $\alpha_0$ given in* (2.4) *obtained on replacing $\theta_{ij}$, $\gamma_{ir}$, and $\lambda_s$ by $\hat{\theta}_{ij}$, $\hat{\gamma}_{ir}$ and $\hat{\lambda}_s$ respectively, then*

MSE $(\hat{Y}_{\hat{\alpha}_0})$ = MSE $(\hat{Y}_{\alpha_0})$ *(to the first order of approximatian)*, *where*

$$E(\hat{\theta}_{ij}) = \theta_{ij}, \qquad j = 1, 2, \ldots, m_i,\ i = 1, 2, \ldots, k$$

$$E(\hat{\gamma}_{ir}) = \gamma_{ir}, \quad r = 1, 2, \ldots, n_i,\ i = 1, 2, \ldots, k$$

and $\quad E(\hat{\lambda}_s) = \lambda_s, \quad s = 1, 2, \ldots, n.$

*Proof* : From (2 3) the mean square error of $\hat{Y}_{\alpha_0}$ is

$$\sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij}(\alpha_o)\, E(e_i\, e_j) \tag{2.5}$$

Let $e_{ij}^{(1)} = (\hat{\theta}_{ij} - \theta_{ij})/\theta_{ij}, \quad j = 1, 2, \ldots, m_i,\ i = 1, 2, \ldots, k$

$e_{ir}^{(2)} = (\hat{\gamma}_{ir} - \gamma_{ir})/\gamma_{ir}, \quad r = 1, 2, \ldots, n_i,\ i = 1, 2, \ldots, k$

$e_s^{(3)} = (\hat{\lambda}_s - \lambda_s)/\lambda_s, \quad s = 1, 2, \ldots, n.$

Using these three equations sets, $\hat{\alpha}_0$ can be written as

$$\hat{\alpha}_0 = \sum_{i=1}^{k} \prod_{j=1}^{m_i} \prod_{s=1}^{n} \prod_{r=1}^{n_i} d_{ij} f_{ir}^{-1} c_s^{-1} \theta_{ij} \gamma_{ir}^{-1} \lambda_s^{-1} \{1 + e_{ij}^{(1)}\} \{2 + e_{ir}^{(2)}\}^{-1} \{1 + e_s^{(3)}\}^{-1}.$$

From this, it is sevident that to the first order of approximation

$$E\{\hat{\alpha}_0^w e_i e_j\} = \alpha_o^w E\{e_i e_j\},\ i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, p,$$

where $w$ is a positive integer. This implies

$$E\{a_{ij}(\hat{\alpha}_0) e_i e_j\} = a_{ij}(\alpha_0) E\{e_i e_j\}, \quad i = 1, 2, \ldots, p, j = 1, 2, \ldots, p.$$

The validity of the statement in the theorem follows from the last identity on appealing to (2.5). Q.E.D.

From the above theorem, we infer that when the parameters in $\alpha_0$ are replaced by their respective unbiased estimators the resulting estimator will have the same mean square error as that of the original estimator.

Sometimes instead of replacing all the unknowns by their respective unbiased estimators, one may be interested in retaining parameters which are functions of the auxiliary variable alone. It can be seen that even in such cases the resulting estimator will have the same mean square error as that of the original estimator. In such cases a result similar to Theorem 2.1 can be proved even when $\alpha$ has a more general form.

We denote by $\theta^{(x)}$, $\gamma^{(x)}$ and $\lambda^{(x)}$ the parameters which are functions of the auxiliary variable $x$ alone and by $\theta$, $\gamma$ and $\lambda$ the parameters which are functions of both $x$ and $y$ or $y$ alone. Let $\varphi\ (\lambda_1^{(x)}, \ldots, \lambda_n^{(x)})$ be any real valued function of $\lambda^{x)}, \ldots, \lambda_n^{(x)}$, which always assumes non-zero value.

THEOREM 2.2 : *If the optimal value of* $\alpha$ *involved in the underlying estimator is of the form*

$$\hat{\alpha}_{00} = \{\varphi(\lambda_1, \ldots, \lambda_n)\}^{-1} \sum_{i=1}^{k} \{ \prod_{j=1}^{m_i'} d_{ij}^{(Y)} \theta_{ij}^{(Y)} \prod_{j=1}^{m_i''} d_{ij}^{(x)} \theta_{ij}^{(x)}\}$$

$$\{\prod_{r=1}^{n_i'} f_{ir}^{(y)} \gamma_{ir}^{(y)} \prod_{r=1}^{n_i''} f_{ir}^{(x)} \gamma_{ir}^{(x)}\}^{-1} \qquad (2.6)$$

then to the first order of approximation MSE $(\hat{Y}_{\hat{\alpha}_{00}})$ = MSE $(\hat{Y}_{\alpha_{00}})$ where $\hat{\alpha}_{00}$ is obtained from $\alpha_{00}$ by replacing $\theta_{ij}^{(y)}$ and $\gamma_{ir}^{(y)}$ by their respective unbiated estimatars.

Proof is similar to Theorem 2.1. Q.E.D.

Results which are similar to Theorems 2.1 and 2.2 can be proved even when the estimator contains more than one unknown value, provided the optimal values of each such quantities are in the form of either $\alpha_0$ or $\alpha_{00}$ as the case may be.

## 3. Discussiou

Srivastava *et al.* [6] have considered the problem of obtaining expressions involving sample quantities for the optimal values of certain estimators. They have suggested the use of different estimators depending on some conditions. As a result of the Section 2 of this paper we infer that the choice of such quantities is immaterial since each of them, when used to replace unknown optimal values in the underlying estimator, yields the same mean square error.

As mentioned in Section 1, Srivastava and Jhajj [5] have proved that in a general class of estimators when the unknown functions are replaced by their 'consistent' estimators the resulting estimator will have the same mean square error as the original estimator, provided certain conditions are satisfied. As a result of Theorem 2.2, we note that one need not always replace unknown population values (particularly functions of $x$) by their unbiased (consistent) estimates. Thus the property proved by Srivastava and Jhajj [5] holds good even when the consistent estimators do not exist provided the optimal values are of the form (2.4) or (2.6). It is of interest to note that the optimal values of unknowns involved in the estimators proposed by Srivastava and Jhajj (1983) are of the form (2.6).

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